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The effect of viscosity on the free stream in transonic flow over a corner point☆

A.N. Bogdanov, V.N. Diyesperov

Moscow, Russia

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Abstract

An asymptotic analysis of the system of Navier–Stokes equations for describing the flow which arises from the subsonic free stream in the neighbourhood of the vertex of a convex corner with curvilinear generatrices is presented for Reynolds numbers approaching infinity. It is assumed that, in limiting non-viscous flow, the subsonic free stream reaches the velocity of sound at the vertex of the corner and, in the first approximation, is described by the Vaglio–Laurin solution. It is shown that the flow can have a different form depending on the value of the pressure gradient, which is formed in the neighbourhood of the corner point. However, irrespective of the steady form of the flow, as a result of the interaction of the Vaglio–Laurin flow with the boundary layer, the latter induces perturbations in the outer flow, which "rounds off" the vertex of the corner when there is a transonic flow around it.

1. Formulation of the problem

The solution of the problem of the local interaction of the boundary layer with outer potential flow in the subsonic neighbourhood of the vertex of a convex corner, formed by a corner profile will be sought in the form of perturbations to the values of the components of the velocity vector at the corner point. We will take as the small parameters the upstream distance from the vertex of the corner and the inverse of the Reynolds number Re, calculated for the critical values of the parameters in the flow. We will assume that, in the limit as $\mathrm{Re} \to \infty$ in a first approximation one obtains Vaglio-Laurin flow,[1] which is formed by a singularity generated by the corner profile. In this case, the boundary conditions in the problem of the flow around the profile with a corner point affect the flow in a small neighbourhood of this point in the following approximations. The interaction of the boundary layer with the outer flow is due to a favourable pressure gradient, which is induced by the Vaglio-Laurin flow. As one approaches the corner point this gradient increases without limit in the form of a power law[2–4] with a coefficient $-3\,\gamma d_\varepsilon/5$ (γ is the ratio of the specific heat capacities and d_ε is a certain constant). Below we will show that the form of the flow in the neighbourhood of the corner point is determined by the value of d_ε, which depends on the boundary conditions outside the neighbourhood of the corner point considered and can only be obtained from the solution of the problem as a whole.

An example of the fact that transonic flow with rarefaction wave due to the corner profile is formed precisely by the boundary conditions and not the singularity, generated by this corner point, is well known.[5] We take half of a plane Laval nozzle and, at the point coinciding with the centre of the nozzle, we produce a discontinuity of the line of

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E-mail address: bogdanov@imec.msu.ru (A.N. Bogdanov).

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symmetry in such a way that a convex corner is obtained. The flow in the channel produced in the neighbourhood of the vertex of the corner differs considerably from Vaglio-Laurin flow, although it also contains a rarefaction wave.

A similar situation also arises in the flow of a viscous incompressible fluid in front of the point of convergence of a free streamline with the surface of a smooth body or a corner point of its generatrix.[6] The structure of such flow was investigated previously in Refs 7,8. The pressure gradient here also increases without limit in accordance with a power law (but a different one) with a coefficient which depends on the constant d_ε. If $d_\varepsilon \ll 1$, the boundary layer in front of the corner point in the first approximation remains linear (Blasius),[9–11] exactly the same as in the free stream. If $d_\varepsilon = O(1)$, then, due to the action of the pressure gradient its own boundary layer is generated with an Ackerberg-type velocity profile.[4,6,11,12]

Experiments show,[13,14] that viscosity plays a considerable role in the formation of the rarefaction wave in transonic flow in the neighbourhood of a corner point. Thanks to the boundary layer the rarefaction waves ceases to be centred and does not issue exactly from the corner point.

2. Outer potential flow

Consider the transonic flow of a perfect gas in the neighbourhood of the corner point of a profile (the point O), obtained as a result of the intersection of two smooth curves AO and OD, where the tangents to them at the point O form a convex corner. We will introduce a Cartesian system of coordinates xOy, the x axis of which coincides with the tangent to the curve AO at the point O. We will use the following notation: Φ is the velocity potential, u and v are its components, p is the pressure, ρ is the density, T is the temperature, a is the velocity of sound and ψ is the stream function. The thermodynamic variables are related by the equation of state of a perfect gas. We will take as the characteristic quantities for all the above parameters their critical values (denoted by an asterisk). Below, all the flow parameters and independent variables are assumed to be dimensionless and are denoted in the previous way.

It is assumed that the irrotational and subsonic free stream along the curve AO reaches the velocity of sound at the point O. In a certain neighbourhood of this point the value of the velocity can be assumed to be a small perturbation with respect to the critical velocity of sound. We will introduce the dimensionless potential of the perturbed velocity such that $\varphi = \Phi - x$, $|u - 1| \ll 1$. In dimensionless variables the gas dynamics equations, describing the transonic flow being considered, reduces to the inhomogeneous Karman equation for the potential of the perturbed velocity

$$\begin{aligned}-(\gamma+1)\varphi_x\varphi_{xx}+\varphi_{yy} &= \left(\frac{\gamma+1}{2}\varphi_x^2+\frac{\gamma-1}{2}\varphi_y^2\right)\varphi_{xx}+2(1+\varphi_x)\varphi_y\varphi_{xy}+\\&+\left(\frac{\gamma+1}{2}\varphi_y^2+\frac{\gamma-1}{2}(2\varphi_x+\varphi_x^2)\right)\varphi_{yy}\end{aligned} \tag{2.1}$$

Karman's equation allows of a class of self-similar solutions[15]

$$\varphi = y^{3n-2}f_0(\xi), \quad \xi = \beta x/y^n, \quad \beta = (1+\gamma)^{-1/3} \tag{2.2}$$

The function $f_0(\xi)$ satisfies the ordinary differential equation[16,17]

$$(n^2\xi^2 - f_0')f_0'' - 5n(n-1)\xi f_0' + 3(n-1)(3n-2)f_0 = 0 \tag{2.3}$$

The solution of Eq. (2.1) can be represented in the form

$$\varphi = y^{3n-2}f_0(\xi) + \sum\varphi_{p_k}; \quad \varphi_{p_k} = y^{p_k}f_{p_k}(\xi) \tag{2.4}$$

One part of the superscripts and subscripts $p_k = 3n - 2 + 2(n - 1)k$ appears in Eq. (2.4) due to the non-linearity of Eq. (2.1) and the bending of the generatrix AO, while the other part is due to the eigenfunctions.

Suppose the equation of the curve AO has the form

$$y = -\chi(n) - (-x)^{4-3/n} + \cdots, \quad x < 0, \quad \chi(n) > 0$$

The solution of the problem of the flow around a corner point, formed by the curves AO and OD, will be sought, in the first approximation, in the class of self-similar solutions of Karman's equation (2.2). It is required that the solution of Eq. (2.3) should satisfy the impermeability condition on AO and convert into a Prandtl - Mayer type solution in the neighbourhood of the point O when $x>0$ and $y\to 0$ (in a centred rarefaction wave[17]). When $n=5/4$ we have $\chi(5/4)=0$. In this case the curve AO becomes a straight line and one obtains Vaglio-Laurin flow.[2,3] When $n\in(5/4,5/3)$ one also obtains flow with wave rarefaction around the corner point, which we will call Vaglio-Laurin type flow,[18] but in this case $\chi(n)\neq 0$. The parametric representation of the solution of Eq. (2.3) when $n=5/4$ has the form[16]

$$f_0(\xi) = d^3(t-1)^{-7/8}(7t^2-140t+160)/21;\quad \xi = d(t-1)^{-5/8}(t-8/5),\quad 1<t<\infty \tag{2.5}$$

where d is a scaling constant (also called the form parameter).[19] This constant depends on the flow conditions outside the neighbourhood of the corner point and can be obtained using the integral law of conservation or from the solution of the problem as a whole. Solution (2.5) shows that the flow considered, in a certain neighbourhood of the vertex of the corner, is formed in a first approximation, exclusively by the singularity formed by the corner around which the flow occurs.

To simplify further calculations we will put $d=3^{-3/8}5^{-1/4}\beta^{-7/8}d_\varepsilon^{5/8}$. The arbitrary constant d_ε can take both values of the order of unity and values $d_\varepsilon\ll 1$.

The solution of Eq. (2.1) for $n=5/4$ is given by expansion (2.4), while the functions $f_{p_k}(\xi)$ satisfy the equations

$$L[f_{p_k}]\equiv\left(\frac{25}{16}\xi^2-f_0'\right)f''_{p_k}-\left[\frac{5}{4}\left(2p_k-\frac{9}{4}\right)\xi+f_0''\right]f'_{p_k}+p_k(p_k-1)f_{p_k} = N_{p_k} \tag{2.6}$$

Changing to the variable $t=t(\xi)$ and substituting $f_{p_k}=(t-1)^{-p_k/2}h_{p_k}(t)$ we convert Eq. (2.6) to an inhomogeneous hypergeometric equation.[17] The general solution of the corresponding homogeneous equation has the form

$$h^k_{p_k} = A^{(s)}_{p_k}F\left(\frac{1}{6}-\frac{2}{3}p_k,-2p_k;\frac{1}{2};1-t\right)+A^{(a)}_{p_k}(t-1)^{1/2}F\left(\frac{1}{2}-2p_k,\frac{2}{3}-\frac{2p_k}{3};\frac{3}{2};1-t\right) \tag{2.7}$$

where $F=F(\cdot;\cdot;\cdot;t)$ is the hypergeometric function.

The fundamental solution, related to the constant $A^{(s)}_{p_k}$, ensures that the impermeability condition is satisfied. We will call it the left-symmetrical solution. We will call the second fundamental solution the left-antisymmetric solution.

Analytic continuation of the solution (2.7) into the region $t\to\infty$ gives

$$\begin{aligned} f^0_{p_k} &= \left\lfloor A^{(s)}_{p_k}B_{11}+A^{(a)}_{p_k}B_{22}\right\rfloor t^{(p_k-1)/6}+\left\lfloor A^{(s)}_{p_k}B_{12}+A^{(a)}_{p_k}B_{21}\right\rfloor t^{3p_k/2}+\ldots = \\ &= D^{(p)}_{p_k}t^{(p_k-1)/6}+D^{(w)}_{p_k}t^{3p_k/2}+\ldots \\ B_{11} &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{1}{6}-\frac{4}{3}p_k\right)}{\Gamma(-2p_k)\Gamma\left(\frac{1}{3}+\frac{2}{3}p_k\right)},\quad B_{12} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{6}+\frac{4}{3}p_k\right)}{\Gamma\left(\frac{1}{6}-\frac{2}{3}p_k\right)\Gamma\left(\frac{1}{2}+2p_k\right)} \\ B_{21} &= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{6}+\frac{4}{3}p_k\right)}{\Gamma\left(\frac{2}{3}-\frac{2}{3}p_k\right)\Gamma(1+2p_k)},\quad B_{22} = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(-\frac{1}{6}-\frac{4}{3}p_k\right)}{\Gamma\left(\frac{1}{2}-2p_k\right)\Gamma\left(\frac{5}{6}+\frac{2}{3}p_k\right)} \end{aligned} \tag{2.8}$$

The perturbed velocity potential $\varphi^0_{p_k}$, corresponding to solutions (2.7) and (2.8), when $x>0$, $y\to 0$, has the form

$$\varphi^0_{p_k}=D^{(p)}_{p_k}y^{(8p_k+1)/9}z^{4(p_k-1)/9}+D^{(w)}_{p_k}z^{4p_k}+\cdots,\quad z=x/y$$

The solution $f^0_{p_k}$ as $\xi\to+\infty$ will be called a Prandtl–Mayer type solution if $D^{(w)}_{p_k}=0$ and a W-type solution if $D^{(p)}_{p_k}=0$. If the left-symmetric solution converts to a Prandtl–Mayer type solution as $\xi\to+\infty$, we will say that it belongs to the class $(S\to P)$, and if it converts into a W-type solution we will say it belongs to the class $(S\to W)$. We define classes for the antisymmetric solution $(A\to P)$ and $(A\to W)$ similarly.

Using the properties of the gamma function, we obtain from relations (2.8)

$$\begin{aligned} &p_k = \frac{6k+1}{4},\ p_k = -\frac{2k+1}{4},\ f_{p_k} \in (S \to P);\ p_k = \frac{k}{2},\ p_k = -\frac{3k+1}{2},\ f_{p_k} \in (S \to W) \\ &p_k = \frac{3k+2}{2},\ p_k = -\frac{k+1}{2},\ f_{p_k} \in (A \to P);\ p_k = \frac{2k+1}{4},\ p_k = -\frac{6k+5}{4},\ f_{p_k} \in (A \to W) \\ &k = 0, 1, 2, \ldots \end{aligned} \tag{2.9}$$

Part of the spectrum of (2.9) when $p_k \to 7/4$ was found when investigating the flow of a perfect gas over a corner point in Ref. 17. Part of the spectrum when $p_k < 7/4$ corresponds to the "inner" solutions, which describe, in particular, slight "rounding off" of the vertex of the corner. The solutions $f_{p_k} \in (S \to P)$ contain arbitrary constants, which reflects the local nature of the problem considered.

We will indicate the asymptotic behaviour of solution (2.4) when $y \to 0$, $x < 0$ and $x > 0$ respectively

$$\varphi = \begin{cases} \frac{5}{7} d_\varepsilon (-x)^{7/5} + O(d_\varepsilon^2 (-x)^{9/5}), & \xi \to -\infty \\ \frac{1}{3} y \beta^3 z^3 + O(d_\varepsilon^{5/3} y^{5/3} z^{1/3}), \quad z = x/y, & \xi \to +\infty \end{cases} \tag{2.10}$$

3. Transonic flow over a corner point by a viscous heat-conducting gas

From formula (2.10) we have

$$u = 1 - d_\varepsilon (-x)^{2/5} + \ldots, \quad \frac{dp}{dx} = -\frac{2}{5} \gamma d_\varepsilon (-x)^{-3/5} + \ldots \tag{3.1}$$

The quantity $dp/dx \to -\infty$ when $x \to 0$. We will investigate the interaction of an infinitely high pressure gradient (3.1) with a boundary layer according to well-known ideas.[7,9,20,21]

We will assume that the surface having the corner point is formally insulated, the coefficient of viscosity and the thermal conductivity are proportional to the temperature, and the Prandtl number Pr = 1. With these conditions the system of equations of the boundary layer possesses an integral, obtained for the first time by Buseman[22],

$$\frac{u^2}{2} + \frac{T}{\gamma - 1} = \frac{1}{(\gamma - 1) R_0}$$

Under the action of an infinitely high pressure gradient (3.1), the flow region in the boundary layer upstream the corner point can be divided into two subregions.[7,12] In the viscous layer close to the wall the flow is mainly produced by the action of viscous stresses; due to the smallness of the velocities and the absence of high temperature gradients the flow, in the first approximation, will be uncompressed. In the bulk of the boundary layer the flow, in the first approximation, is vortex flow, inviscid and described by Euler's equations. The nature of the interaction depends on the coefficient d_ε, the value of which is formed by the problem as a whole. For small values $d_\varepsilon \ll 1$ in the boundary layer in the first approximation the linear "Blasius" profile of the longitudinal velocity in the lower viscous sublayer is preserved. When $d_\varepsilon = O(1)$ in the neighbourhood of the corner point its non-linear velocity profile is formed.[4] The description of the form of the flow as a function of the value of d_ε in the case of an incompressible fluid is well known.[6] Below, for convenience of the calculations, we put $d_\varepsilon = \varepsilon d_0$ in formula (3.1), where $d_0 = O(1)$, $\varepsilon \in (0,1]$.

4. Interaction of the boundary layer with Vaglio-Laurin flow in the case when $d_\varepsilon \ll 1$

We will assume that $\varepsilon \ll 1$, while the velocity profile of the boundary layer in the first approximation is linear with a coefficient of proportionality λ.[8] In the lower viscous sublayer the solution of the boundary-layer equations for the stream function ψ will be sought in the form

$$\psi = R_0\left\{\frac{1}{2}\lambda Y^2 + (-x)^{4/3}f_0(\eta) + \varepsilon(-x)^{2/5}F_0(\eta) + \dots\right\}$$
$$f_0(\eta) = \frac{1}{6}\lambda^3(\gamma-1)R_0, \quad \eta = Y(-x)^{-1/3}, \quad Y = y\mathrm{Re}^{1/2} \tag{4.1}$$

Here R_0 is the value of the density at the point O while Y is the coordinate of the boundary layer in the transverse direction.

To determine $F_0(\eta)$, the following linear equation is obtained

$$\Phi''' - \frac{1}{3}\zeta^2\Phi'' + \frac{2}{5}\zeta\Phi' - \frac{2}{5}\Phi = -1; \quad F_0 = \frac{2d_0}{5\lambda R_0}\Phi(\zeta), \quad \zeta = (\lambda R_0^2)^{1/3}\eta \tag{4.2}$$

From the solution of Eq. (4.2) it is required that

$$\Phi(0) = \Phi'(0) = 0$$

while when $\zeta \to \infty$ it should not contain exponentially increasing terms. The successive substitutions

$$\Phi = \zeta f, \quad \frac{df}{d\zeta} = \zeta^{-2}g(\zeta), \quad \sigma = \frac{1}{9}\zeta^3$$

reduce Eq. (4.2) to a degenerate hypergeometric equation[10]

$$\sigma g'' + \left(\frac{1}{3} - \sigma\right)g' + \frac{2}{5}g = -1$$

Its solution, which satisfies the boundary conditions, has the form

$$\Phi = \zeta\int_0^\zeta\left[K_1\Psi\left(-\frac{2}{5}, \frac{1}{3}; \sigma\right) - \frac{5}{2}\right]\zeta^{-2}d\zeta; \quad K_1 = \frac{5\Gamma\left(\frac{4}{15}\right)}{2\Gamma\left(\frac{2}{3}\right)} \tag{4.3}$$

where $\Psi(\cdot,\cdot,\sigma)$ is the Tricomi function. From (4.3) we obtain

$$\Phi = 3^{-4/5}K_2\zeta^2 + O(\zeta^3); \quad K_2 = K_1\Gamma\left(-\frac{2}{3}\right)\left[\Gamma\left(-\frac{2}{5}\right)\right]^{-1} \text{ as } \zeta \to 0$$
$$\Phi = 3^{-4/5}\cdot 5K_1\zeta^{6/5} + D_1\zeta + \frac{5}{2} + O(\zeta^{-9/5}) \text{ as } \zeta \to \infty \tag{4.4}$$
$$D_1 = \int_0^\infty\left[K_1\Psi\left(-\frac{2}{5}, \frac{1}{3}, \sigma\right) - \frac{5}{2} - 3^{-4/5}K_1\zeta^{6/5}\right]\zeta^{-2}d\zeta$$

The surface friction τ_w, calculated using solution (4.3), has the form

$$\tau_w = \lambda + (4/5)3^{-4/3}R_0^{1/3}K_2 d_\varepsilon\lambda^{-1/3}(-x)^{-4/15} + \dots \tag{4.5}$$

In the bulk of the boundary layer, according to expansions (4.4), we will seek a solution in the form

$$u = U(Y) - xu_{01}(Y) + \varepsilon\lfloor u_{10}(Y) + (-x)^{1/15}u_{11}(Y) + (-x)^{2/5}u_{12}(Y)\rfloor + \dots$$
$$V = V_0(Y) - xV_{01}(Y) + \varepsilon\lfloor(-x)^{-14/15}V_{11}(Y) + (-x)^{-2/3}V_{12}(Y)\rfloor + \dots \tag{4.6}$$
$$\rho = R(Y) - x\rho_{01}(Y) + \varepsilon\lfloor\rho_{10}(Y) + (-x)^{1/15}\rho_{11}(Y) + (-x)^{2/5}\rho_{12}(Y)\rfloor + \dots$$

The functions $U(Y)$ and $u_{10}(Y)$ are arbitrary, while $u_{1i}(Y)$, $V_{1i}(Y)$, $\rho_{1i}(Y)$ $(i \geq 1)$ satisfy ordinary differential equations, as a result of the solution of which we obtain

$$\begin{aligned}
&V_{11} = C_{11}U, \quad u_{11} = 15C_{11}U', \quad \rho_{11} = (\gamma-1)Uu_{11}B^{-2} \\
&V_{12} = \frac{2}{5}d_0U\int_Y^\infty [R^{-1}U^{-2} - 1 - (\lambda Y)^{-2}R_0^{-1}]dY + 2d_0U(5\lambda^2R_0Y)^{-1} + C_{12}U \\
&u_{12} = d_0U'\int_Y^\infty [R^{-1}U^{-2} - 1 - (\lambda Y)^{-2}R_0^{-1}]dY + d_0[U'\lambda^{-2}Y^{-1}R_0^{-1} - U^{-1}R^{-1}] + \frac{5}{2}C_{12}U' \\
&\rho_{12} = (\gamma-1)Uu_{12}B^{-2} + \gamma d_0B^{-1}, \quad B = R_0^{-1} - \frac{\gamma-1}{2}U^2
\end{aligned} \tag{4.7}$$

Expansions (4.6) are matched with the expansions in the viscous sublayer; the details of the matching are omitted. As a result of the matching, we obtain

$$C_{11} = \frac{2}{75}R_0^{-1/3}d_0\lambda^{-5/3}D_1, \quad C_{12} = \frac{2}{5}d_0\int_0^\infty [R^{-1}U^{-2} - 1 - (\lambda Y)^{-2}R_0^{-1}]dY$$

If we require that $[U(Y) \to 1,\ U'(Y) \to \infty]$ when $Y \to \infty$, we obtain from expansions (4.6) and relations (4.7)

$$u = 1 - d_\varepsilon(-x)^{2/5} + \ldots, \ \rho = 1 + d_\varepsilon(-x)^{2/5} + \ldots, \ V = \varepsilon[C_{11}(-x)^{-14/15} + C_{12}(-x)^{-3/5} + \ldots] \tag{4.8}$$

Hence, as a result of the action of an infinitely high pressure gradient (3.1) on the boundary layer, the latter induces perturbations (4.8) in the outer potential flow. The first term in the third formula of (4.8) is due to displacement thickness of the boundary layer, while the second term is due to the pressure. Expansion (4.8) has meaning if the second term is considerably less than the first. This condition leads to the estimate $(-x) \ll \lambda^{-5}$. On the other hand, it follows from relation (4.5) that $\lambda^{-5}d_\varepsilon^{15/4} \ll (-x)$. Combining both estimates, we obtain that the expansion constructed makes sense if

$$\lambda^{-5}d_\varepsilon^{15/4} \ll (-x) \ll \lambda^{-5} \tag{4.9}$$

since $d_\varepsilon \ll 1$, $x < 0$.

The solution in the outer potential region, induced by the perturbed boundary layer, will be sought in the form

$$\varphi = y^{7/4}f_0(\xi) + \mathrm{Re}^{-1/2}[y^{-1/6}f_{-1/6}(\xi) + y^{1/4}f_{1/4}(\xi) + \ldots] \tag{4.10}$$

We will consider the solution of the problem in the first approximation. The number $p_k = -1/6$, as can be seen from (2.9), is not spectral. The function $f_{-1/6}(\xi)$ has the form

$$f_{-1/6}(\xi) = D_{-1/6}^{(S)}(t-1)^{1/12}F\left(\frac{5}{18}, \frac{1}{3}; \frac{1}{2}; 1-t\right) + D_{-1/6}^{(A)}(t-1)^{7/12}F\left(\frac{5}{6}, \frac{7}{9}; \frac{3}{2}; 1-t\right)$$

Matching expansions (4.8) and (4.10), we obtain

$$D_{-1/6}^{(A)} = 2 \cdot 3^{-19/12}5^{-5/6}\beta^{7/4}R_0^{-1/3}D_1\lambda^{-5/3}d_\varepsilon^{5/12}$$

Analytic continuation of $f_{-1/6}(\xi)$ into the region $\zeta \to \infty$, $x > 0$, $y \to 0$ is given by formula (2.8). If we require that a solution of the Prandtl–Mayer centred rarefaction wave type should be obtained, it is necessary to put

$$D_{-1/6}^{(S)} = -\frac{B_{21}\left(-\frac{1}{6}\right)}{B_{12}\left(-\frac{1}{6}\right)} D_{-1/6}^{(A)}$$
$$\varphi = \frac{5}{7} d_\varepsilon (-x)^{-7/5} + \mathrm{Re}^{-1/2}\left[-\frac{15}{12}\omega_{-1/6}^{(S)} \varepsilon^{1/2} \lambda^{-5/3} (-x)^{-2/15}\right] + \dots \tag{4.11}$$
$$\omega_{-1/6}^{(S)} = 4 \cdot 3^{-5/2} \cdot 5^{-2} \beta^{3/2} R_0^{-1/3}(0) D_1 d_0^{1/2} B_{21} B_{12}^{-1}$$

Expansion inside the boundary layer and in the outer potential region must be valid for values of the variable x of one order. A comparison of expansions (4.11) and (4.5) shows that this condition is satisfied if the parameters λ, d_ε and Re are connected by the relation

$$\lambda d_\varepsilon^{-25/24} \mathrm{Re}^{-1/12} = K \tag{4.12}$$

The relation between the parameters λ, d_ε and Re was obtained earlier[6] for the case of the flow of an incompressible fluid over a corner point.

In the next approximation it is necessary to take into account the effect of the second term in the square brackets in (4.10) (with index $p_2 = 1/4$), which belongs to the spectral set (2.9). We obtain that one of the fundamental solutions belongs to the class ($A \to W$), and other belongs to the class ($S \to P$). The general solution has the form

$$f_{1/4} = D_{1/4}^{(S)}(t-1)^{-1/8} + D_{1/4}^{(A)}(t-1)^{3/8} \tag{4.13}$$

The second term in solution (4.13) is due to the effect of the perturbed boundary layer, and when $\xi \to \infty$ ($t \to \infty$) it converts into a W type solution, which describes the flow around a curved surface.

Matching with expansion (4.8), generally speaking, determines the non-zero constant

$$D_{1/4}^{(A)} = 3^{-3/8} \cdot 5^{3/4} \beta^{9/8} C_{12} d_\varepsilon^{5/8}$$

Hence, the effect of viscosity manifests itself, in particular, in that the boundary layer necessarily "rounds off" the corner over which flow occurs. Since the fundamental solutions, which occur in the general solution (4.13), correspond to the classes ($A \to W$) and ($S \to P$), the constant $D_{1/4}^{(S)}$ remains undetermined. The indeterminant form of $f_{1/4}^{(S)}$ is a consequence of the locality of the solution of the problem. When $x > 0$, $y \to 0$ we have

$$\varphi_{1/4}(x, y) = D_{1/4}^{(P)} y^{1/3} z^{-1/3} + D_{1/4}^{(W)} z + \dots; \quad z = x/y \tag{4.14}$$

The behaviour of the potential $\varphi_{1/4}$ described above denotes that if the flow is considered in two approximations, then for expansions (4.10) one cannot establish the condition for a transition to a Prandtl–Mayer type solution when $x > 0$, $y \to 0$, since, generally speaking, the constant $D_{1/4}^{(W)} \neq 0$ in Eq. (4.14).

5. The interaction of the boundary layer with the Vaglio-Laurin flow when $d_\varepsilon = O(1)$

We will consider another type of flow in the boundary layer, assuming that $d_\varepsilon = O(1)$. We will take into account that, in this case, due to the action of the infinite pressure gradient in the boundary layer an Ackerberg type velocity profile,[4,11] is obtained, which produces a non-linear longitudinal-velocity profile in the boundary layer when $y \to 0$ ($x \to 0-$).

In a thin layer near the wall $Y \sim (-x)^{2/5}$ the solution will be sought in the form[4,11]

$$\bar{\psi} = (-x)^{3/5} R_0 F(\eta) + \dots, \quad \eta = (-x)^{-2/5} Y, \quad \rho = R_0 + (-x)^{2/5} \rho_0(\eta) + \dots$$
$$p = 1 + \gamma d_\varepsilon (-x)^{2/5} + \dots \tag{5.1}$$

The function $F(\eta)$ satisfies the equation

$$-R_0^{-2}F''' + \frac{3}{5}FF'' - \frac{1}{5}F'^2 = \frac{2}{5}d_\varepsilon R_0^{-1} \tag{5.2}$$

The transformation

$$\eta = 5^{1/2}R_0^{-3/4}(2d_\varepsilon)^{-1/4}\zeta, \quad F(\eta) = 5^{1/2}R_0^{-5/4}(2d_\varepsilon)^{1/4}\Phi(\zeta)$$

reduces Eq. (5.2) to canonical form

$$3\Phi\Phi'' - \Phi'^2 - \Phi''' = 1 \tag{5.3}$$

It is required to obtain a solution of Eq. (5.3), which satisfies the condition

$$\Phi = \Phi' = 0 \quad \text{for} \quad \zeta(0)$$

and which, when $\zeta \to \infty$, has the asymptotic behaviour

$$\Phi(\zeta) = b_0\zeta^{3/2} + b_{00}\zeta^{1/2}\ln\zeta + b_0\zeta^{1/2} + \cdots, \quad b_{00} = -(3b_0)^{-1}, \quad F = B_0\eta^{3/2} + B_{00}\eta^{1/2}\ln\eta + B_{01}\eta^{1/2} + \cdots$$

It was proved in Ref. 23 that the problem in question has a unique solution. The function ρ_0 is found from the Buseman integral[22]

$$\rho_0 = \gamma R_0 d_\varepsilon + \frac{\gamma - 1}{2}R_0^2F^2$$

The expansions of the flow parameters in the main part of the boundary layer will have the form

$$\begin{aligned} &u = U(Y) + (-x)^{2/5}[u_0(Y)\ln(-x) + u_1(Y)] + \ldots, \quad V = (-x)^{3/5}[V_0(Y)\ln(-x) + V_1(Y)] + \ldots \\ &\rho = R(Y) + (-x)^{2/5}[\rho_0(Y)\ln(-x) + \rho_1(Y)] + \ldots, \quad p = 1 + \gamma d_\varepsilon(-x)^{2/5} + \ldots \end{aligned} \tag{5.4}$$

The functions u_i, V_i, ρ_i $(i = 0,1)$ satisfy ordinary differential equations. Matching expansions (5.1) and (5.4) when $Y \to 0$ gives[4,11]

$$U(Y) = \frac{3}{2}B_0Y^{1/2} + \cdots, \quad R(Y) = R_0 + \frac{9}{8}(\gamma - 1)R_0^2B_0^2Y + \cdots$$

Hence, when $d_\varepsilon = O(1)$, a non-linear Ackerberg type profile of the longitudinal component of the velocity is generated.

Under the conditions when

$$U(Y) \to 1, \quad R(Y) \to 1 \quad (U'(Y) \to 0,\ R'(Y) \to 0) \quad \text{as} \quad Y \to \infty$$

expansions (5.4) for u, ρ and p are automatically matched with the expansions in the inner potential region. For the components of the velocity V we obtain

$$\begin{aligned} &V = [A_0\ln(-x) + A_1](-x)^{-3/5} + \ldots \\ &A_0 = \frac{16}{225}d_\varepsilon R_0^{-1}B_0^{-2}, \quad A_1 = \frac{4}{15}(B_{01} - B_{00})B_0^{-1} - \frac{2}{5}d_\varepsilon\omega_1 \\ &\omega_1 = \int_0^\infty (R^{-1} - U^2)[U^{-2} - 4(3B_0)^{-2}Y^{-1}]dY - 4(3B_0)^{-2}\int_0^\infty \frac{d}{dY}(R^{-1} - U^2)\ln Y dY \end{aligned} \tag{5.5}$$

Hence it follows that, as a result of the action of the pressure gradient (3.1) on the boundary layer, the latter induces additional perturbations, proportional to $\mathrm{Re}^{-1/2}$ in the outer potential flow. We will seek the expansion of the potential of the perturbed velocity in the form

$$\varphi = y^{7/4}f_0(\xi) + \ldots + \mathrm{Re}^{-1/2}y^{1/4}[\chi_{1/4}(\xi)\ln(y) + f_{1/4}(\xi)] + \ldots \tag{5.6}$$

The functions $\chi_{1/4}, f_{1/4}$ satisfy the following equations (the symbol L corresponds to the left part of Eq. (2.6))

$$L\left\lfloor \chi_{1/4} \right\rfloor = 0, \quad L[f_{1/4}] = \frac{5}{2}\xi\chi'_{1/4} + \frac{1}{2}\chi_{1/4}$$

from which we have

$$\begin{aligned}\chi_{1/4} &= B_{1/4}^{(S)}(t-1)^{-1/8} + B_{1/4}^{(A)}(t-1)^{3/8} \\ f_{1/4} &= \left[\frac{1}{6}B_{1/4}^{(S)}\ln[t^4(t-1)^{-3}] + 4B_{1/4}^{(A)}\operatorname{arctg}\sqrt{t-1} - B_{1/4}^{(S)} + D_{1/4}^{(S)}\right](t-1)^{-1/8} + \\ &+ \left[\frac{1}{2}B_{1/4}^{(A)}\ln[t^4(t-1)^{-3}] - \frac{4}{3}B_{1/4}^{(S)}\operatorname{arctg}\sqrt{t-1} - 3B_{1/4}^{(A)} + D_{1/4}^{(A)}\right](t-1)^{3/8}\end{aligned} \tag{5.7}$$

It can be seen from conditions (2.9) that the solutions $\chi_{1/4}$, $f_{1/4}$ can be subdivided into the direct sum of two fundamental solutions of the class ($S \to P$) and ($A \to W$). Because of this the constants $B_{1/4}^{(S)}$ and $D_{1/4}^{(S)}$ remain arbitrary and can only be determined when solving the problem as a whole. We will assume that $B_{1/4}^{(S)}$ and $D_{1/4}^{(S)}$ are of the same order as $B_{1/4}^{(A)}$ and $D_{1/4}^{(A)}$. The latter are expressed in terms of the constants A_0 and A_1 by matching expansions (5.5) and (5.6):

$$A_0 = 4 \cdot 5^{-8/5}\beta^{-3/5}(3d_\varepsilon)^{3/5}B_{1/4}^{(A)}, \quad A_1 = D_{1/4}^{(A)}$$

A comparison of the terms in expansion (5.6) with respect to order of magnitude gives the characteristic dimensions

$$x^* = O\left\lfloor Re^{-5/12}\ln^{5/6}Re \right\rfloor, \quad y^* = O[x^{*4/5}]$$

at which the expansions obtained in the subsonic region lose validity.

As a result of the interaction of the outer Vaglio-Laurin potential flow with the boundary layer the corner is "rounded off" more strongly, which in the first approximation flow of the simple-wave type occurs. The strong rounding off of the corner may be related, in particular, to the formation of a developed local separating bubble in a certain region downstream the corner point.[14]

Note that in the supersonic region $z = x/y = O(1)$, $y \to 0$ the expansion of the total potential is sought as the solution of Euler's equations in the form[11]

$$\Phi = yg_0(z) + y^{5/3}g_1(z) + \mathrm{Re}^{-1/2}\{W(z) + y^{1/3}[P_0(z)\ln(y) + P_1(z)]\} + \ldots \tag{5.8}$$

The functions $g_0(z)$ and $g_1(z)$ were found previously in Refs 3 and 19. The first term in the braces describes a simple wave and is due to "rounding off" of the vertex of the corner, while the second describes the centred rarefaction wave. The functions $P_0(z)$ and $P_1(z)$ can be determined from the solutions of first-order ordinary differential equations. The behaviour of the function $W(z)$, like the behaviour of the functions $P_0(z)$ and $P_1(z)$, when $z \to 0$, are determined from the condition for expansions (5.6) and (5.8) to be matched. For the remaining values of z, as in the case when $d_\varepsilon \ll 1$, the function $W(z)$ is arbitrary.

Expansion (5.8) ceases to be valid when $x^* = O(\mathrm{Re}^{-3/10})$, $y^* = O(\mathrm{Re}^{-3/10})$. These expansions are characteristic for transonic flows of a heat-conducting gas.[24]

The solutions constructed for $d_\varepsilon = O(1)$ and for $d_\varepsilon \ll 1$ show that the solution generated in the neighbourhood of the corner point depends on the problem considered. A similar situation is observed in the flow of an incompressible fluid over a corner point.[6] The locality of the problem considered above also manifests itself in the fact that unknown constants and arbitrary functions arise in the solutions constructed. The occurrence of an arbitrary function $W(z)$, which describes a simple rarefaction wave, is due to the fact that in the supersonic neighbourhood of the corner the flow may acquire a different form depending on the problem considered. In particular, the form of the flow in the supersonic region depends very much on the shape of the generatrix of the corner OD.

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References

1. Vaglio-Laurin R. Transonic rotational flow over a convex corner. *J. Fluid Mech.* 1960;**9**(1):81–103.
2. Yesin AI, Chernov IA. The construction of a uniformly valid solution in the neighbourhood of a sonic corner point of the generatrix of a solid of revolution. In *Aerodinamika. Saratov: Izd Sarat Univ* 1975;**4**(7):36–46.
3. Boichenko VS, Lifshits YuB. Transonic flow over a convex corner. *Uchen Zap TsAGI* 1976;**7**(2):8–16.
4. Diyesperov VN. Transonic gas flow over a convex corner. *Prikl Mat Mekh* 1980;**44**(1):60–8.
5. Diyesperov VN. Transonic rarefaction flow in the neighbourhood of a convex corner. *Prikl Mat Mekh* 1981;**45**(4):651–61.
6. Bogdanov YeV, Ryzhov OS. Laminar preseparation flow. *Prikl Mat Mekh* 1986;**50**(3):394–402.
7. Ackerberg RC. Boundary-layer separation at a free streamline. Pt 1. Two-dimensional flows. *J Fluid Mech* 1970;**44**(Pt. 2):211–25.
8. Sychev, editor. *The Asymptotic Theory of Separation Flows*. Moscow: Nauka; 1987.
9. Sychev VV. Laminar separation. *Izv Akad Nauk SSSR MZhG* 1972;**3**:47–59.
10. Messiter AF, Enlow RL. A model for laminar boundary layer flow near a separation point. *J SIAM Appl Math* 1973;**25**(2):44–54.
11. Diesperov VN. Viscous transonic flow in the vicinity of a corner point. Symposium transsonicum III IUTAM symposium. Göttingen, 1988. Ed. J. Zierep, H. Oertel. Berlin, Hiedelberg: Springer, 1989. P. 199–205.
12. Ruban AI. Laminar separation from a corner point of a solid surface. *Uchen Zap TsAGI* 1974;**5**(2):44–54.
13. Sears WR, editor. *General theory of high speed aerodynamics*. Princeton: Univ. Press; 1954. p. 666c.
14. Bryson AE, Jun E. An experimental investigation of transonic flow past two-dimensional wedge and circular are sections using a Mach-Zehnder interferometer. *NACA Rep* 1952:1094.
15. Guderley G. *Theorie Schallnaher Strömungen*. Berlin: Springer; 1957.
16. Fal'kovich SV, Chernov IA. Supersonic gas flow around a solid of revolution. *Prikl Mat Mekh* 1964;**28**(2):280–4.
17. Yesin AI, Chernov IA. The problem of local flow in the vicinity of a convex corner. In *Aerodinamika. Saratov: Izd Sarat Univ* 1978;**6**(9):17–32.
18. Diyesperov VN. Some solutions of Karman's equation describing transonic flow over a corner point with a curvilinear generatrix. *Prikl Mat Mekh* 1994;**58**(6):68–77.
19. Yesin AI, Fal'kovich SV, Chernov IA. *Transonic flow of a perfect gas in the vicinity of a corner point. Aerodinamika. Saratov Izd Sarat Univ*, 1976;5(**8**). pp. 3–10.
20. Neiland VYa. Theory of the separation of a laminar boundary layer in supersonic flow. *Izv Akad Nauk SSSR MZhG* 1969;**4**:53–7.
21. Stewartson K, Williams PG. Self-induced separation. *Proc Roy Soc London Ser A* 1969;**312**(1509):181–206.
22. Pai SI. *Introduction to the Theory of Compression Flow*. Princeton: Van Nostrand; 1959.
23. McLeod JB. The existence and uniqueness of a similarity solution arising from separation at a free stream line. *Quart J Math* 1972;**23**(2):63–77.
24. Messiter AF, Feo A, Melnik RE. Shock-wave strength for separation of a laminar boundary layer at transonic speed. *AIAA Journal* 1971;**9**(6):1197–8.

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